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Phil. Trans. R. Soc. Lond. A 1994 **346**, 173-190
doi: 10.1098/rsta.1994.0017

Attractors for the modifications of the three-dimensional Navier–Stokes equations

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The modifications of the three-dimensional Navier–Stokes equations, which I suggested earlier for the description of viscous fluid flows with large gradients of velocities, are considered. It is proved that the first initial-boundary value problem for these equations in any bounded three-dimensional domain has a compact minimal global B-attractor. Some properties of the attractor are established.

1. Introduction

The equations which we shall consider have the form

$$v_{it}(x,t)+\sum_{k=1}^{3} v_k(x,t)\, v_{ik}(x,t)-\sum_{k=1}^{3}\frac{\partial}{\partial x_k}T_{ik}(\hat{v}(x,t))=-q_{x_i}(x,t)+f_i(x,t),\quad i=1,2,3,\tag{1}$$

$$\operatorname{div} v(x,t)=0,\quad x\subset\Omega\subset R^3,\tag{2}$$

where $v(t):\Omega\to\mathbb{R}^3$ and $q(t):\Omega\to\mathbb{R}^1$ are unknown functions; $v_i(x,t)$, $i=1,2,3$, are components of velocity vector $v(x,t)$, $\hat{v}(x,t)=(v_{ik}(x,t))$ $(i,k=1,2,3)$, a tensor with components $v_{ik}(x,t)=v_{ix_k}(x,t)+v_{kx_i}(x,t)$, $v_{ix_k}(x,t)=\partial_{x_k}v_i(x,t)$, $T(\hat{v}(x,t))=(T_{ik}(\hat{v}(x,t)))$ $(i,k=1,2,3)$ – a stress tensor which we shall describe later, $f_i(x,t)$ – the components of the external forces $f(x,t)$ which are considered known.

For the Navier–Stokes equations

$$T_{ik}(\hat{v})=\nu_0 v_{ik},\quad \nu_0=\text{const.}>0.$$

For the modified Navier–Stokes equations that I suggested in the middle 1960s ([1], see also [2, 4]) T_{ik} are continuous functions of the components of $\hat{v}$ satisfying the following conditions:

(i) $|T_{ik}(\hat{v})|\leqslant c_1(1+|\hat{v}|^{2\mu})\,|\hat{v}|$, $\mu>0$, where $|\hat{v}|=(\sum_{i,k=1}^{3}v_{ik}^2)^{\frac{1}{2}}$, $c_1\in\mathbb{R}^+\equiv[0,\infty)$;

(ii) $T_{ik}(\hat{v})\,v_{ik}\geqslant\nu_0\hat{v}^2+\nu_1\hat{v}^{2+2\mu}$, where ν_0 and ν_1 are positive constants and $\hat{v}^2=|\hat{v}|^2$, $\hat{v}^{2+2\mu}=|\hat{v}|^{2+2\mu}$,

(iii) for arbitrary smooth solenoidal vector-fields v' and v'' which are equal on the boundary $\partial\Omega$ the following inequality holds:

$$\int_\Omega[T_{ik}(\hat{v}')-T_{ik}(\hat{v}'')]\,(v'_{ik}-v''_{ik})\,\mathrm{d}x\geqslant\nu_2\int_\Omega\sum_{i,k=1}^{3}(v'_{ik}-v''_{ik})^2\,\mathrm{d}x,\quad \nu_2=\text{const.}>0.$$

It was proved ([3]) that a global unique solvability of the principal boundary value problem for the system (1), (2) in any $\Omega\subset\mathbb{R}^3$ takes place if only $\mu\geqslant\frac{1}{4}$. It is true also for the two-dimensional problems if only $\mu\geqslant 0$ but we restrict ourselves with the more interesting three-dimensional problems.

Phil. Trans. R. Soc. Lond. A (1994) **346**, 173–190
Printed in Great Britain

In this paper we shall study a special case of tensors T_{ik}:

$$T_{ik}(\hat{v}) = \beta(\hat{v}^2)\, v_{ik}, \quad i, k = 1, 2, 3, \tag{3_1}$$

that was singled out in [3] (see also [2, 4]) as most interesting from physical point of view. These tensors satisfy the Stokes axioms. They satisfy also all conditions (i)–(iii) if function $\beta: R^+ \to R^+$ is continuous and submits to inequalities:

$$\nu_0 + \nu_1 \tau^\mu \leqslant \beta(\tau) \leqslant \nu_3 + \nu_4 \tau^\mu, \quad \nu_k = \text{const.} > 0, \tag{3_2}$$

and

$$\beta'(\tau) \geqslant 0. \tag{3_3}$$

We shall suppose here that inequalities (3_2) are fulfilled with $\mu \geqslant \frac{1}{4}$ and $\beta(\cdot)$ is an absolutely continuous function with

$$\beta' \in L_{\infty, \text{loc}}. \tag{3_4}$$

For the sake of simplicity we restrict ourselves by the case of homogeneous boundary condition

$$v|_{\partial\Omega} = 0. \tag{4_1}$$

We study the dependence of the solutions v to the system (1), (2), (4_1) on initial data

$$v|_{t=0} = \phi \tag{4_2}$$

and the behaviour of the solutions when $t \to +\infty$. We denote solutions of (1), (2), (4_1), (4_2) by the symbol $v(t)$ or $v(t, \phi)$ having in mind their dependence on $x \in \Omega$.

Let us introduce the function

$$\mathscr{B}(\tau) = \int_0^\tau \beta(s)\, \mathrm{d}s. \tag{3_5}$$

It submits to the inequalities

$$\nu_0 \tau + \nu_1 (1+\mu)^{-1} \tau^{1+\mu} \leqslant \mathscr{B}(\tau) \leqslant \nu_3 \tau + \nu_4 (1+\mu)^{-1} \tau^{1+\mu}, \quad \mu \geqslant \tfrac{1}{4}, \tag{3_6}$$

$$[T_{ik}(\hat{v}') - T_{ik}(\hat{v}'')]\,(v'_{ik} - v''_{ik}) \geqslant \nu_0 \sum_{i,k=1}^{3} (v'_{ik} - v''_{ik})^2, \tag{3_7}$$

and

$$\frac{\partial}{\partial v_{jl}} T_{ik}(\hat{v})\, \xi_{jl}\, \xi_{ik} = \beta(\hat{v}^2) \sum_{i,k=1}^{3} \xi_{ik}^2 + 2\beta'(\hat{v}^2) \left(\sum_{i,k=1}^{3} v_{ik}\, \xi_{ik} \right)^2 \geqslant \nu_0 \sum_{i,k=1}^{3} \xi_{ik}^2 \tag{3_8}$$

with arbitrary $\hat{v}$, $\hat{v}'$, $\hat{v}''$ and ξ_{ik}.

I also proposed ([1]) the system (1), (2) with

$$T_{ik}(\hat{v}(x,t)) = \left[\nu_0 + \nu_1 \int_\Omega \hat{v}^2(x,t)\, \mathrm{d}x \right] v_{ik}(x), \quad \nu_i > 0. \tag{5}$$

It is globally uniquely solvable under the conditions (4_1) and (4_2) ([2]), and the problem (1), (2), (4_1) with such T_{ik} has a compact minimal global B-attractor $\mathfrak{M}$ ([5]). This attractor has the same properties as the attractor $\mathfrak{M}$ for the two-dimensional Navier–Stokes equations proved in [6] and [7] (see also [8–10]). The methods of [5] are closed to the methods of [6]. In particular, the existence of a compact $\mathfrak{M}$ is based on the compactness of the solution operators $V_t, t > 0$, of the investigated problem.

But the method of proving the compactness of V_t, $t > 0$, suggested in [6] (in all subsequent studies of attractors for the Navier–Stokes equations was used just this method, see [11–13]) has some 'weak sides': (i) it is not applicable to the most part of known approximations of the Navier–Stokes equations (including the Galerkin–Faedo approximations calculated with the help of coordinate functions that do not coincide with the eigenfunctions of the Stokes operator $\tilde{\Delta}$), (ii) it is making use the linearity or 'almost' linearity (as in (1) with $T_{i,k}$ from (5)) of the principal part of the system under investigation and (iii) it requires the C^2-smoothness of $\partial\Omega$. At last it is (ii) that makes the method of [6] just mentioned not applicable to the problem considered in this paper.

We use for it a comparatively new method suggested by us in 1991 for the Navier–Stokes equations ([14], see also [15, 16]). At the end of [16] the possibility to apply the approach of [14] to the problem under discussion was noted.

2. 'Energy' estimates and absorbing balls

Let $v(\cdot,t,\phi) \equiv v(t,\phi)$ be a solution of (1), (2), (4_1), (4_2) with (T_{ik}) satisfying (3_k), $k = 1,\dots,4$, with $\mu \geqslant \frac{1}{4}$. As a phase space we choose the Hilbert space H_0 which is the closure in the norm of Hilbert space $L_2(\Omega;R^3)$ of the set

$$\dot{J}^\infty(\Omega) = \{u \parallel u \in C^\infty(\Omega), \operatorname{div} u = 0, \operatorname{supp} u \quad \text{is compact in } \Omega\}.$$

(In [4] we used for H_0 the symbol $\mathring{J}(\Omega)$ and have proved that the orthogonal complement to $\mathring{J}(\Omega)$ in $L_2(\Omega;R^3)$ is the set of gradient's fields.) The inner product and the norm in $L_2(\Omega;R^3)$ and in H_0 we shall denote by $(\cdot,\cdot)$ and $\|\cdot\|$ correspondingly. Let us introduce two other Hilbert spaces: the space H_1 which is the closure of the set $\dot{J}^\infty(\Omega)$ in the norm of Dirichlet integral

$$\|u\|_1 = \|u_x\| \equiv \left(\int_\Omega \sum_{i,k} u^2_{ix_k}(x)\,\mathrm{d}x\right)^{\frac{1}{2}}$$

and the space H_{-1}, which is dual to H_1 relative to H_0. The norm in H_{-1} is defined, as usually, by

$$\|u\|_{-1} = \sup_{\phi\in H_1} \frac{|(u,\phi)|}{\|\phi_x\|}.$$

The norm in Banach space $L_m(\Omega;R^n)$ is determined by the equality

$$\|u\|_{m,\Omega} = \left(\int_\Omega |u(x)|^m\,\mathrm{d}x\right)^{1/m},$$

where

$$-u(x)| = \left(\sum_{i=1}^n u_i^2(x)\right)^{\frac{1}{2}}.$$

(Recall that the norm $\|u\|_{2,\Omega}$ we have designated for simplicity by $\|u\|$.)

Besides, for $u \in W^1_m(\Omega;R^3)$ the symbol $\hat{u}$ means the tensor (u_{ik}), $i,k = 1,2,3$, with elements

$$u_{ik} = u_{ix_k} + u_{kx_i}, \quad |\hat{u}| = \left(\sum_{i,k=1}^3 u^2_{ik}\right)^{\frac{1}{2}}$$

and
$$\|\hat{u}\|_{m,\Omega} = \|\,|\hat{u}|\,\|_{m,\Omega}.$$

We shall investigate the set of all solutions of the problem (1), (2), (4_1), (4_2) with

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a fixed $f \in H_{-1}$ independent from t and any $\phi \in H_0$. The estimates given below are easily generalized for f depending on t.

The energy relation for our problem has the form

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2 + \frac{1}{2}\int_\Omega T_{ik}(\hat{v}(x,t))\, v_{ik}(x,t)\,\mathrm{d}x = (f, v(t)). \tag{6}$$

We get it if we multiply (1) by $v_i(x,t)$, take the sum over $i = 1,2,3$, integrate over $x \in \Omega$ and transform the result by the integration by parts using the relations (2) and (4_1). We have used also the symmetry of (T_{ik}) and (v_{ik}).

Due to (3_2) from (6) we can come to conclusions:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2 + \nu_0\|\hat{v}(t)\|^2 + \nu_1\|\hat{v}(t)\|^{2+2\mu}_{2+2\mu,\Omega} \leqslant 2(f, v(t)) \leqslant 2\|f\|_{-1}\|v_x(t)\|. \tag{7}$$

It is easy to see that

$$\|\hat{u}\|^2 = 2\|u_x\|^2 \quad \text{for any} \quad u \in H_1. \tag{8}$$

Therefore from (7) follows the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2 + \nu_0\|v_x(t)\|^2 + \nu_1\|\hat{v}(t)\|^{2+2\mu}_{2+2\mu,\Omega} \leqslant \nu_0^{-1}\|f\|^2_{-1}. \tag{9}$$

Integration of (9) gives the estimate

$$\|v(t)\|^2 + \nu_0\int_0^t \|v_x(\tau)\|^2\,\mathrm{d}\tau + \nu_1\int_0^t \|\hat{v}(\tau)\|^{2+2\mu}_{2+2\mu,\Omega}\,\mathrm{d}\tau$$
$$\leqslant \|v(0)\|^2 + \nu_0^{-1}t\|f\|^2_{-1} \equiv \Phi_1(t, \|v(0)\|). \tag{10}$$

If Ω is bounded domain then for any $u \in H_1$

$$\|u_x\|^2 \geqslant \lambda_1\|u\|^2, \tag{11}$$

where $-\lambda_1 \equiv -\lambda_1(\Omega)$ is the first eigenvalue of the Stokes operator $\tilde{\Delta} = P_{H_0}\Delta$ under the boundary condition (4_1) (see [4], chapter II). In this case from (9) follows the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2 + \nu_0\lambda_1\|v(t)\|^2 \leqslant \nu_0^{-1}\|f\|^2_{-1}. \tag{12}$$

Its integration gives the estimate

$$\|v(t)\|^2 \leqslant \|v(0)\|^2\exp(-\nu_0\lambda_1 t) + \|f\|^2_{-1}(\lambda_1\nu_0^2)^{-1}[1-\exp(-\nu_0\lambda_1 t)]. \tag{13}$$

From (13) we can conclude that any ball

$$B_R(H_0) \equiv \{u \,\|\, u \in H_0, \|u\| \leqslant R\} \tag{14_1}$$

with

$$R > R_0 \equiv \lambda_1^{-\frac{1}{2}}v_0^{-1}\|f\|_{-1} \tag{14_2}$$

is a B-absorbing set for our problem. It means that for any $R_1 > 0$ exists some $t_1(R,R_1) \in \mathbb{R}^+$ such that all solutions $v(t)$ of the problem (1), (2), (4_1) with $\|v(0)\| \leqslant R_1$ will have $\|v(t)\| \leqslant R$ for all $t \geqslant t_1(R,R_1)$. Besides, if $\|v(0)\| \leqslant R$ and $R \geqslant R_0$ then $\|v(t)\| \leqslant R$ for all $t \in \mathbb{R}^+$. This is why the attractor $\mathfrak{M}$ of our problem (if it exists) lies in the ball $B_{R_0}(H_0)$ and to find $\mathfrak{M}$ and investigate it we can choose for the phase space any ball $B_R(H_0)$ with $R \geqslant R_0$.

3. Estimates of some strong norms

To get farther estimates we multiply (1) by $v_{it}(x,t)$, sum over $i = 1,2,3$ and integrate over $x \in \Omega$. Some elementary transformations of the result give the relation

$$\|v_t(t)\|^2 + \frac{1}{4}\int_\Omega \beta(\hat{v}^2(x,t))\,\partial_t\,\hat{v}^2(x,t)\,\mathrm{d}x = -(v_k(t)\,v_{ik}(t), v_{it}(t)) + (f, v_t(t)),$$

which can be written in the form

$$\|v_t(t)\|^2 + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathscr{B}(\hat{v}^2(x,t))\,\mathrm{d}x = -(v_k(t)\,v_{ik}(t), v_{it}(t)) + (f, v_t(t)), \tag{15}$$

where $\mathscr{B}(\cdot)$ is determined in (3_5).

We shall use now and later on the inequality

$$\|u\|_{p,\Omega} \leqslant \gamma_p\,\|u\|^{1-\alpha}\,\|u_x\|^\alpha, \quad \alpha = \tfrac{3}{2} - 3/p, \quad p \in [2, 6], \tag{16}$$

which holds for any $u \in \mathring{W}_2^1(\Omega)$ and any $\Omega \subset \mathbb{R}^3$ (see [17]). Due to [16] we can derive from [15] the inequalities

$$\|v_t(t)\|^2 + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathscr{B}(\hat{v}^2(x,t))\,\mathrm{d}x \leqslant \|\hat{v}(t)\|_{2+2\mu,\Omega}\,\|v(t)\|_{p,\Omega}, \|v_t(t)\|_{p,\Omega} + \|f\|_{-1}\,\|v_{xt}(t)\|$$

$$\leqslant \gamma_p^2\,\|\hat{v}(t)\|_{2+2\mu,\Omega}\,\|v(t)\|^{1-\alpha}\,\|v_x(t)\|^\alpha\,\|v_t(t)\|^{1-\alpha}\,\|v_{xt}\|^\alpha + \|f\|_{-1}\,\|v_{xt}(t)\|, \tag{17}$$

where $p = 4(1+\mu)(1+2\mu)^{-1}$ and $\alpha = 3\cdot 4^{-1}(1+\mu)^{-1}$.

Let us consider one more relation: the result of multiplication by v_{it} of the equation (1), differentiated in t and the subsequent summation over $i = 1,2,3$ and integration over $x \in \Omega$. It can be cast in the form

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_t(t)\|^2 &+ \frac{1}{2}\int_\Omega \frac{\partial T_{ik}(\hat{v}(x,t))}{\partial v_{jl}}\,v_{jlt}(x,t)\,v_{ikt}(x,t)\,\mathrm{d}x \\ &= -(v_k(t)\,v_{ikt}(t) + v_{kt}(t)\,v_{ik}(t), v_{it}(t)) \\ &= -\frac{1}{2}\int_\Omega v_{ki}(x,t)\,v_{kt}(x,t)\,v_{it}(x,t)\,\mathrm{d}x.\end{aligned} \tag{18}$$

Now we use (3_8) for to estimate from below the left-hand side of (18) and Hölder, Young's inequalities and (16) for to majorize the right-hand side of (18). Namely

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v_t(t)\|^2 + 2\nu_0\,\|v_{xt}(t)\|^2 &\leqslant \|\hat{v}(t)\|_{2+2\mu,\Omega}\,\|v_t(t)\|^2_{p,\Omega} \\ &\leqslant \gamma_p^2\,\|\hat{v}(t)\|_{2+2\mu,\Omega}\,\|v_t(t)\|^{2(1-\alpha)}\,\|v_{xt}(t)\|^{2\alpha} \\ &\leqslant \alpha\nu_0\,\|v_{xt}(t)\|^2 + (1-\alpha)\,\gamma_p^{2/(1-\alpha)}\,\nu_0^{-\alpha/(1-\alpha)}\,\|\hat{v}(t)\|^{1/(1-\alpha)}_{2+2\mu,\Omega}\,\|v_t(t)\|^2\end{aligned}$$

with the same p and α as above in (17). From this relation follows the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v_t(t)\|^2 + \nu_0(2-\alpha)\,\|v_{xt}(t)\|^2 \leqslant c_1\,\|\hat{v}(t)\|^{1/(1-\alpha)}_{2+2\mu,\Omega}\,\|v_t(t)\|^2 \tag{19}$$

with $c_1 = (1-\alpha)\,\gamma_p^{2/(1-\alpha)}\,\nu_0^{-\alpha/(1-\alpha)}$.

Our condition $\mu \geqslant \frac{1}{4}$ is equivalent to the inequality $(1-\alpha)^{-1} = 4(1+\mu)/(1+4\mu) \leqslant 2+2\mu$. Due to this we can use the estimate (10) and get

$$\int_0^t \|\hat{v}(\tau)\|^{1/(1-\alpha)}_{2+2\mu,\Omega}\,\mathrm{d}\tau \leqslant t^{\mu_1}\left(\int_0^t \|\hat{v}(\tau)\|^{2+2\mu}_{2+2\mu,\Omega}\,\mathrm{d}\tau\right)^{(2+2\mu)^{-1}(1-\alpha)^{-1}} \leqslant \Phi_2(t, \|v(0)\|). \tag{20}$$

Here $\mu_1 = [(2+2\mu)(1-\alpha)-1]\,(2+2\mu)^{-1}(1-\alpha)^{-1} \geqslant 0$ and the majorant $\Phi_2(\cdot)$ as well as subsequent majorants $\Phi_k(\cdot)$ are non-negative, continuous and non-decreasing function of indicated arguments. We do not show the dependence of Φ_k on the constants entering in our conditions (3_k) and on $\|f\|_{-1}$ (and on $\lambda_1 = \lambda_1(\Omega)$ for the case of bounded Ω). All Φ_k can be easily calculated explicitly.

If $\|v_t(0)\| < \infty$ then the integration of (19) gives the estimate

$$\|v_t(t)\|^2 + \int_0^t \|v_{x\tau}(\tau)\|^2\,\mathrm{d}\tau \leqslant \Phi_3(t, \|v(0)\|, \|v_t(0)\|). \tag{21}$$

But we shall not use this estimate in the following consideration and instead of (21) we shall estimate $\|v_t(t)\|$ and $\int_\epsilon^t \|v_{x\tau}(\tau)\|^2\,\mathrm{d}\tau$ for $t \geqslant \epsilon > 0$ using only the norm $\|v(0)\|$.

For this purpose we consider the sum of (17) multiplied by $4t$ and (19) multiplied by t^2. It can be written in the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left(t\int_\Omega \mathscr{B}(\hat{v}^2(x,t))\,\mathrm{d}x + t^2\,\|v_t(t)\|^2\right) + 2t\,\|v_t(t)\|^2\\ &+\nu_0(2-\alpha)\,t^2\,\|v_{xt}(t)\|^2 \leqslant \int_\Omega \mathscr{B}(\hat{v}^2(x,t))\,\mathrm{d}x\\ &+4\gamma_p^2\,\|\hat{v}(t)\|_{2+2\mu,\Omega}\,\|v(t)\|^{1-\alpha}\,\|v_x(t)\|^{\alpha}(t\,\|v_t(t)\|)^{1-\alpha}(t\,\|v_{xt}(t)\|)^{\alpha}\\ &+4\,\|f\|_{-1}(t\,\|v_{xt}(t)\|) + c_1\,\|\hat{v}(t)\|^{1/(1-\alpha)}_{2+2\mu,\Omega}(t^2\,\|v_t(t)\|^2).\end{aligned} \tag{22}$$

Let us represent the second member of the right-hand side of (22) as the product of the following three factors

$$j(t) = [4\gamma_p^2\,\nu_0^{-\alpha/2}\,\|\hat{v}(t)\|^{\frac{1}{2}}_{2+2\mu,\Omega}\,\|v(t)\|^{1-\alpha}\,\|v_x(t)\|^{\alpha}][\|\hat{v}(t)\|^{\frac{1}{2}}_{2+2\mu,\Omega}\,(t\,\|v_t(t)\|)^{1-\alpha}]\,[\sqrt{\nu_0}\,t\,\|v_{xt}(t)\|]^{\alpha}$$

and majorize it using the Young's inequality with the powers $p_1 = 2, p_2 = 2/(1-\alpha)$ and $p_3 = 2/\alpha$ (so $1/p_1 + 1/p_2 + 1/p_3 = 1$):

$$\begin{aligned}j(t) \leqslant 8\gamma_p^4\,\nu_0^{-\alpha}\,\|\hat{v}(t)\|_{2+2\mu,\Omega}\,\|v(t)\|^{2(1-\alpha)}\,\|v_x(t)\|^{2\alpha}&\\ +\tfrac{1}{2}(1-\alpha)\,\|\hat{v}(t)\|^{1/(1-\alpha)}_{2+2\mu,\Omega}(t^2\,\|v_t(t)\|^2) + \tfrac{1}{2}\alpha\,\nu_0\,t^2\,\|v_{xt}(t)\|^2.&\end{aligned} \tag{23_1}$$

To the third member of the right-hand side of (22) we apply Cauchy inequality in the form

$$4\|f\|_{-1}(t\|v_{xt}(t)\|) \leqslant \tfrac{1}{10}\nu_0\,t^2\,\|v_{xt}(t)\|^2 + 40\nu_0^{-1}\,\|f\|^2_{-1}. \tag{23_2}$$

Now we estimate the right-hand side of (22) using inequalities (23_k), reduce the similar terms and replace in the left-hand side of the resulting inequality the coefficient $\nu_0(2-\alpha-\frac{1}{2}\alpha-\frac{1}{10})$ of $t^2\,\|v_{xt}(t)\|^2$ by 1 (it is possible because $\alpha \leqslant \frac{3}{5}$). After that we get the inequality

$$\begin{aligned}\frac{\mathrm{d}y(t)}{\mathrm{d}t} &+ 2t\,\|v_t(t)\|^2 + \nu_0\,t^2\,\|v_{xt}(t)\|^2 \leqslant \int_\Omega \mathscr{B}(\hat{v}(x,t))\,\mathrm{d}x\\ &+c_2(t)\,\|\hat{v}(t)\|_{2+2\mu,\Omega}\,\|v_x(t)\|^{2\alpha} + (c_1+\tfrac{1}{2}(1-\alpha))\,\|\hat{v}(t)\|^{1/(1-\alpha)}_{2+2\mu,\Omega}(t^2\,\|v_t(t)\|^2)\\ &+40\nu_0^{-1}\,\|f\|^2_{-1} = c_3(t) + c_4(t)\,(t^2\,\|v_t(t)\|^2) \leqslant c_3(t) + c_4(t)\,y(t),\end{aligned} \tag{24}$$

where

$$y(t) = t\int_{\Omega} \mathscr{B}(\hat{v}(x,t))\,\mathrm{d}x + t^2\,\|v_t(t)\|^2, \tag{25_1}$$

$$c_2(t) = 8\gamma_p^4\,\nu_0^{-\alpha}\,\|v(t)\|^{2(1-\alpha)}, \tag{25_2}$$

$$c_3(t) = \int_{\Omega} \mathscr{B}(\hat{v}(x,t))\,\mathrm{d}x + c_2(t)\,\|\hat{v}(t)\|_{2+2\mu,\Omega}\,\|v_x(t)\|^{2\alpha} + 40\nu_0^{-1}\,\|f\|_{-1}^2, \tag{25_3}$$

and

$$c_4(t) = (c_1 + \tfrac{1}{2}(1-\alpha))\,\|\hat{v}(t)\|_{2+2\mu,\,\Omega}^{1/(1-\alpha)}. \tag{25_4}$$

The inequality (10) gives majorants for

$$\sup_{t\in[0,\tau]} c_2(t), \quad \|c_3\|_{1,(0,\tau)} \quad \text{and} \quad \|c_4\|_{1,(0,\tau)}$$

(for $\forall\tau\in\mathbb{R}^+$) as the powers $(1+2\alpha)$ and $(1-\alpha)^{-1}$ do not exceed the number $2+2\mu$. (For the case of bounded Ω it is reasonable to use (13).) These majorants have the form $\Phi(t,\|v(0)\|)$. In virtue of this fact we can majorize $y(t)$ and the integral $\int_0^t\cdot$ of the left-hand side of (24) by a $\Phi(t,\|v(0)\|)$ and recalling (3_6) come to the estimates

$$\nu_0\,\|\hat{v}(t)\|^2 + \nu_1(1+\mu)^{-1}\,\|\hat{v}(t)\|_{2+2\mu,\Omega}^{2+2\mu} + t\,\|v_t(t)\|^2 \leqslant t^{-1}\,\Phi_4(t,\|v(0)\|) \tag{26}$$

and

$$\int_0^t (2\tau\,\|v_\tau(\tau)\|^2 + \nu_0\,\tau^2\,\|v_{x\tau}(\tau)\|^2)\,\mathrm{d}\tau \leqslant \Phi_5(t,\|v(0)\|). \tag{27}$$

4. Continuity of solution operators V_t

As was said in the Introduction the problem (1), (2), (4_1), (4_2) is uniquely solvable for all $t\in\mathbb{R}^+$ for any $\phi\in H_0$ if T_{ik} satisfy the conditions (i)–(iii) with $\mu\geqslant\frac{1}{4}$. Its solutions $v(t,\phi)$ are elements of $\mathbb{C}(R^+,H_0)$ ([3]). Though in [3] we supposed that $f\in L_{2,1}(Q_\tau)$, $\forall\tau\in R^+$, (where $Q_\tau = \Omega\times(0,\tau)$), the case with $f\in L_2((0,\tau),H_{-1})$, $\forall\tau\in R^+$, and the more so the case with f belonging to H_{-1} and independent on t can be considered in the similar way.

In the last case the solution operators

$$V_t\colon \phi \longrightarrow v(t,\phi) \equiv V_t(\phi) \tag{28}$$

have the semi-group property

$$V_{t_1}V_{t_2} = V_{t_1+t_2}, \quad \forall t_1,t_2\in\mathbb{R}^+.$$

The estimate (10) guarantees the boundedness of operators V_t, $t\in R^+$, in H_0. Their continuity is proved with the help of the same inequality as the uniqueness of solutions. Namely, let us take two solutions $v'(t) = v(t,\phi')$ and $v''(t) = v(t,\phi'')$ of our problem and consider their difference $u(t) = v'(t) - v''(t)$. The latter is a solution of the linear problem

$$u_{it}(t) + v_k'(t)\,u_{ik}(t) + u_k(t)\,v_{ik}''(t) - \frac{\partial}{\partial x_k}[T_{ik}(\hat{v}'(t)) - T_{ik}(\hat{v}''(t))] = -p_{x_i}, \quad i=1,2,3, \tag{29_i}$$

$$\operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \quad u(0) = \phi' - \phi''. \tag{29_4}$$

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Let us consider the sum over $i = 1,2,3$ of the inner products (29_i) and $v_i(t)$. The result can be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+\frac{1}{2}\int_\Omega [T_{ik}(\hat{v}'(x,t))-T_{ik}(\hat{v}''(x,t))]\,u_{ik}(x,t)\,\mathrm{d}x$$
$$=-\int_\Omega [\omega_k(x,t)\,u_{ik}(x,t)\,u_i(x,t)+u_k(x,t)\,\omega_{ik}(x,t)\,u_i(x,t)]\,\mathrm{d}x$$
$$=-\tfrac{1}{2}(\omega_{ki}(t),u_k(t)\,u_i(t)), \tag{30}$$

where $\omega(x,t)=\frac{1}{2}(v'(x,t)+v''(x,t))$. Using (3_7), (16) and a Young's inequality we can come from (30) to the following conclusions

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+2\nu_0\,\|u_x(t)\|^2\leqslant\|\hat{\omega}(t)\|_{2+2\mu,\Omega}\,\|u(t)\|^2_{2p,\Omega}$$
$$\leqslant\|\hat{\omega}(t)\|_{2+2\mu,\Omega}\,\gamma^2_{2p}\,\|u(t)\|^{2(1-\alpha)}\,(\nu_0\,\|u_x(t)\|^2)^\alpha\,\nu_0^{-\alpha}$$
$$\leqslant\alpha\nu_0\,\|u_x(t)\|^2+(1-\alpha)\,(\gamma^2_{2p}\,\nu_0^{-\alpha})^{1/(1-\alpha)}\,\|u(t)\|^2\,\|\hat{\omega}(t)\|^{1/(1-\alpha)}_{2+2\mu,\Omega}, \tag{31}$$

where p and α are the same as in (17).

The reduction of similar members gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+\nu_0(2-\alpha)\,\|u_x(t)\|^2\leqslant c_5(t)\,\|u(t)\|^2 \tag{32}$$

with

$$c_5(t)=(1-\alpha)(\gamma^2_{2p}\,\nu_0^{-\alpha})^{1/(1-\alpha)}\,\|\hat{\omega}(t)\|^{1/(1-\alpha)}_{2+2\mu,\Omega}. \tag{33}$$

The estimate (10) and $1/(1-\alpha)\leqslant 2+2\mu$ give a majorant Φ_6 for $\|c_5\|_{1,(0,\tau)}$:

$$\int_0^t c_5(\tau)\,\mathrm{d}\tau\leqslant\Phi_6(t,\|\phi'\|,\|\phi''\|) \tag{34}$$

and therefore

$$\|u(t)\|=\|V_t(\phi')-V_t(\phi'')\|\leqslant\|\phi'-\phi''\|\exp\{\tfrac{1}{2}\Phi_6(t,\|\phi'\|,\|\phi''\|)\}. \tag{35}$$

The estimate (35) ensures the continuity of V_t in H_0 which is uniform on any bounded set of H_0.

5. The finiteness of the number of determining modes

We can do one more interesting conclusion from (32) if Ω is bounded.

Suppose that the solutions $v'(t)=v(t,\phi')$ and $v''(t)=v(t,\phi'')$, $t\in R^+$ can be prolonged for $t\in R^-=(-\infty,0]$ as solutions of (1), (2), (4_1) on all axis $t\in R^1$ and

$$\sup_{t\in R^1}(\|v(t,\phi')\|,\|v(t,\phi'')\|)<\infty.$$

(Later on, in §6, we shall see that such properties have only the solutions, lying on the attractor $\mathfrak{M}$). Let us suppose also that their orthogonal projections $P^m v(t,\phi')$ and $P^m v(t,\phi'')$ on the subspace $P^m H_0$ spanned on m first eigenfunctions of the Stokes operator Δ (under the boundary condition (4_1)) coincide for all $t\in R^1$, i.e.

$$P^m u(t)\equiv P^m v(t,\phi')-P^m v(t,\phi'')\equiv 0,\quad \forall t\in R^1. \tag{36}$$

Then

$$u(t)\equiv 0,\quad \forall t\in R^1, \tag{37}$$

if only m is sufficiently large.

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Let us prove this fact. Due to (36) $u(t)$ for any $t \in R^1$ belongs to the subspace $Q^m H_0$ which is orthogonal to $P^m H_0$ and therefore

$$\|u_x(t)\|^2 \geqslant \lambda_{m+1} \|u(t)\|^2, \quad \forall t \in R^1, \tag{38}$$

where λ_{m+1} is $(m+1)$th eigenvalue of $-\tilde{\Delta}$ (see [4], chapter II). Using (38) we deduce from (32) the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^2 \leqslant -[\nu_0(2-\alpha)\lambda_{m+1} - c_5(t)] \|u(t)\|^2. \tag{39}$$

Its integration gives

$$\|u(t)\|^2 \leqslant \|u(\tau)\|^2 \exp\left\{-\nu_0(2-\alpha)\lambda_{m+1}(t-\tau) + \int_\tau^t c_5(s)\,\mathrm{d}s\right\} \tag{40}$$

for any $\tau < t$. If we can find a natural number m such that for any $t \in R^1$

$$\overline{\lim_{\tau \to -\infty}} (t-\tau)^{-1} \int_\tau^t c_5(s)\,\mathrm{d}s < \nu_0(2-\alpha)\lambda_{m+1}, \tag{41}$$

then from (40) will follow (37).

To satisfy (41) we use the information (7) on the solutions $v'(t)$ and $v''(t)$. From (7) follow the inequalities

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|v'(t)\|^2 + \|v''(t)\|^2) + \nu_1(\|\hat{v}'(t)\|^{2+2\mu}_{2+2\mu,\Omega} + \|\hat{v}''(t)\|^{2+2\mu}_{2+2\mu,\Omega}) \leqslant \nu_0^{-1} \|f\|^2_{-1}$$

and

$$\int_\tau^t (\|\hat{v}'(s)\|^{2+2\mu}_{2+2\mu,\Omega} + \|\hat{v}''(s)\|^{2+2\mu}_{2+2\mu,\Omega})\,\mathrm{d}s \leqslant \nu_1^{-1}(\|\hat{v}(\tau)\|^2 + v''(\tau)\|^2)$$
$$+ \nu_0^{-1}\nu_1^{-1}(t-\tau)\|f\|^2_{-1}, \quad -\infty \to \tau < t < \infty. \tag{42}$$

Now we use (42) to find a majorant for the integral

$$j(t,\tau) = \int_\tau^t \|\hat{\omega}(s)\|^{1/(1-\alpha)}_{2+2\mu,\Omega}\,\mathrm{d}s$$

appearing in the left-hand side of (41) (see (33) determining $c_5(t)$). Namely,

$$j(t,\tau) = \int_\tau^t \left\|\frac{\hat{v}(s)+\hat{v}''(s)}{2}\right\|^{1/(1-\alpha)}_{2+2\mu,\Omega} \mathrm{d}s \leqslant \left(\int_\tau^t \left\|\frac{\hat{v}'(s)+\hat{v}''(s)}{2}\right\|^{2+2\mu}_{2+2\mu,\Omega} \mathrm{d}s\right)^{2/(1+4\mu)}$$
$$\times (t-\tau)^{(4\mu-1)/(4\mu+1)} \leqslant [\nu_1^{-1}(\|v'(\tau)\|^2 + \|v''(\tau)\|^2) + \nu_0^{-1}\nu_1^{-1}(t-\tau)\|f\|^2_{-1}]^{2/(1+4\mu)}$$
$$\times (t-\tau)^{(4\mu-1)/(4\mu+1)} \leqslant \nu_1^{-1}(\|v'(\tau)\|^2 + \|v''(\tau)\|^2)^{2/(1+4\mu)}(t-\tau)^{(4\mu-1)/(4\mu+1)}$$
$$+ (\nu_0^{-1}\nu_1^{-1}\|f\|^2_{-1})^{2/(1+4\mu)}(t-\tau)$$

and therefore

$$\overline{\lim_{\tau \to -\infty}} (t-\tau)^{-1} \int_\tau^t c_5(s)\,\mathrm{d}s \leqslant (1-\alpha)(\gamma^2_{2p}\nu_0^{-\alpha})^{1/(1+\alpha)}(\nu_0^{-1}\nu_1^{-1}\|f\|^2_{-1})^{2/(1+4\mu)}. \tag{43}$$

We shall satisfy the condition (41) if we take m so large that

$$\lambda_{m+1} > [(1-\alpha)/\nu_0(2-\alpha)](\gamma^2_{2p}\nu_0^{-\alpha})^{1/(1+\alpha)}(\nu_0^{-1}\nu_1^{-1}\|f\|^2_{-1})^{2/(1+4\mu)}. \tag{44}$$

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The smallest natural number m for which (37) follows from (36) is given the name of *the number of determining modes* for a solution $v'(t)$ or for a bounded set composed by 'full trajectories' of the problem (i.e. solutions of the problem determined for all $t \in R^1$).

The number m for which the inequality (44) is fulfilled is a majorant for any bounded set in H_0 composed by full trajectories. Such maximal set is the minimal global B-attractor $\mathfrak{M}$ of the problem (1), (2), (4_1).

6. Minimal global *B*-attractor

Here we consider the problem (1), (2), (4_1) in a bounded domain $\Omega \subset \mathbb{R}^3$ choosing H_0 as the phase space.

Let us introduce one more space $\mathscr{H}$. It is the closure of $\dot{J}^\infty(\Omega)$ in the norm

$$\|u\|_{\mathscr{H}} \equiv \|\hat{u}\|_{2+2\mu,\,\Omega}. \tag{45_1}$$

This norm is equivalent on $\dot{J}^\infty(\Omega)$ to the norm

$$\|u\|_{\mathscr{H}} \equiv \|\hat{u}\|_{2+2\mu,\,\Omega} + \|u_x\|_{2,\,\Omega} + \|u\|_{2,\,\Omega}, \tag{45_2}$$

since for any $u \in \dot{J}^\infty(\Omega)$

$$\|u\|_{2,\,\Omega} \leqslant c(\Omega)\|u_x\|_{2,\,\Omega}, \quad \|u_x\|_{2,\,\Omega} = 1/\sqrt{2}\,\|\hat{u}\|_{2,\,\Omega}, \tag{45_3}$$

$$\|\hat{u}\|_{2,\,\Omega} \leqslant |\Omega|^{\mu/(2+\mu)}\,\|\hat{u}\|_{2+2\mu,\,\Omega}. \tag{45_4}$$

Note that

$$\|u_x\|_{2+2\mu,\,\Omega} \leqslant c\,\|\hat{u}\|_{2+2\mu,\,\Omega} \tag{46}$$

for any u belonging to $\mathring{W}^1_{2+2\mu}(\Omega)$ (see for example [18]). But for our purposes it is sufficiently to know only (45_k) and do not use (46).

The space $\mathscr{H}$ is a separable Banach space (see Appendix A) and therefore in $\mathscr{H}$ there are coordinate (fundamental) systems $\{a^l(x)\}_{l=1}^\infty$. Let us take one of them. Without loss of generality we suppose that the system $\{a^l(x)\}$ is normalized in H_0, i.e. $(a^k, a^l) = \delta^l_k$.

In [3] we required that $\{a^l\}_{l=1}^\infty$ form a basis in $\mathscr{H}$ but in reality we used only the fact that $\{a^l\}_{l=1}^\infty$ is a fundamental system in $\mathscr{H}$.

So, according to [3], the solution $v(t, \phi)$ of the problem (1), (2), (4_1), (4_2) is found as a limit of the Galerkin–Faedo approximations

$$v^m(x,t) = \sum_{l=1}^m c_l^m(t)\,a^l(x), \tag{47_1}$$

which are determined by the following system of ordinary differential equations

$$\int_\Omega [v^m_{it}(x,t)\,a^l_i(x) + v^m_k(x,t)\,v^m_{ik}(x,t)\,a^l_i(x) + \tfrac{1}{2}T_{ik}(\hat{v}^m(x,t))\,a^l_{ik}(x)]\,\mathrm{d}x$$

$$= \int_\Omega f_i(x)\,a^l_i(x)\,\mathrm{d}x, \quad l = 1, \ldots, m, \tag{47_2}$$

with initial data

$$c_l^m(0) = (\phi, a^l), \quad l = 1, \ldots, m. \tag{47_3}$$

In (47_2) $a^l_i(x)$, $i = 1, 2, 3$, are components of $a^l(x)$ and $a^l_{ik}(x) = a^l_{ix_k}(x) + a^l_{kx_i}(x)$. It is easy to see that for solutions v^m of (47_2), (47_3) all *a priori* estimates which we have proved in the preceding sections do hold. On the base of these estimates we can conclude the unique solvability of the systems (47_2), (47_3) for all $t \in \mathbb{R}^+$ and prove a convergence of v^m to the solution v of the problem (1), (2), (4_1), (4_2). We shall not describe here how it is done as the proofs are similar to the arguments of [3]. Let us note only that now we have some additional estimates for v^m (the estimates of §3) which give corresponding information about exact solutions of our problem and which facilitate the limit procedure. In this way the theorem is proved:

Theorem 1. *The problem* (1), (2), (4_1), (4_2) *in an arbitrary bounded domain* $\Omega \subset \mathbb{R}^3$ *with* $f \in H_{-1}$ *and* $\phi \in H_0$ *has unique solution* $v(t,\phi)$ *with the following properties*:

$$v \in \mathbb{C}(R^+, H_0) \cap L_2((0,\tau), H_1) \cap L_{2+2\mu}((0,\tau), \mathscr{H}) \cap L_\infty((\epsilon,\tau), \mathscr{H})$$

and

$$v_t \in L_\infty((\epsilon,\tau), H_0) \cap L_2((\epsilon,\tau), H_1)$$

for arbitrary $0 < \epsilon < \tau < \infty$. *The solution operators* (28) *form the semi-group* $\{V_t, t \in \mathbb{R}^+, H_0\}$ *of the first class. It has a bounded B-absorbing set* (14_1) *with* $R > R_0$.

Let us recall that in the term 'semi-group' the property of continuity of $V_t : H_0 \rightarrow H_0$ is included and the belonging of a semi-group to the first class means that operators V_t, $t > 0$, are compact. The continuity of V_t for the problem (1), (2), (4_1) follows from (35) and the compactness of V_t, $t > 0$, follows from (26) and from the fact that the balls of $\mathscr{H}$ are precompacts in H_0.

According to the theory of attractors for the semi-groups of the first class ([8–10]) we can deduce from the Theorem 1 and estimates (26), (27) the following

Theorem 2. *The semi-group* $\{V_t, t \in \mathbb{R}^+, H_0\}$ *has the minimal global B-attractor* $\mathfrak{M}$. *It is a compact connected invariant subset of* H_0 *bounded in* $\mathscr{H}$:

$$\sup_{\phi \in \mathfrak{M}} \{\|\hat{\phi}\|_{2+2\mu,\Omega}\} \leqslant \Phi_7(\|f\|_{-1}). \tag{48}$$

Attractor $\mathfrak{M}$ *is maximal among bounded invariant subsets of* H_0. *Each solution* $v(t,\phi)$ *with* $\phi \in \mathfrak{M}$ *can be extended as a solution* $v(t,\phi)$, $t \in \mathbb{R}^1$, *of the system* (1), (2), (4_1) *for all* $t \in (-\infty, 0)$ *and*

$$\sup_{\phi \in \mathfrak{M}} \sup_{t \in \mathbb{R}^1} \left\{ \|v_t(t,\phi)\|, \int_t^{t+1} \|v_{x\tau}(\tau,\phi)\|^2 \, d\tau \right\} \leqslant \Phi_8(\|f\|_{-1}). \tag{49}$$

The number of determined modes on $\mathfrak{M}$ *does not exceed the number* m *for which holds the inequality* (44).

The approximations described in §6 are globally stable in the following sense:

Theorem 3. *Let* $\{a^l\}_{l=1}^{\infty}$ *be a coordinate system in the space* $\mathscr{H}$ *and* $v^m(t)$, $m = 1, 2, \ldots$, *be the Galerkin–Faedo approximations* (47_1) *satisfying the system* (47_2), (47_3). *For any natural* m *are determined the solution operators*

$$V_t^m : \phi \longrightarrow v^m(t,\phi), \quad t \in \mathbb{R}^+,$$

acting in the subspace $H_0^m = span\ \{a^1, \ldots, a^m\}$ *of* H_0. *The operators form the semi-group* $\{V_t^m, t \in \mathbb{R}^+, H_0^m\}$. *It has a minimal global B-attractor* $\mathfrak{M}^m$. *There are common majorants*

$$\sup_{m \in N^+} \sup_{\phi \in \mathfrak{M}^m} \{\|\hat{\phi}\|_{2+2\mu,\Omega}\} \leqslant \Phi_7(\|f\|_{-1}) \tag{50}$$

and

$$\sup_{m \in N^+} \sup_{\phi \in \mathfrak{M}^m} \sup_{t \in \mathbb{R}^1} \left\{ \|v_t^m(t,\phi)\|, \int_t^{t+1} \|v_{x\tau}(\tau,\phi)\|^2 \, d\tau \right\} \leqslant \Phi_8(\|f\|_{-1}). \tag{51}$$

For any $\epsilon > 0$ *there is a number* $m(\epsilon) \in N^+$ *such that*

$$\mathfrak{M}^m \subset O_\epsilon(\mathfrak{M}) \quad \textit{for all} \quad m \geqslant m(\epsilon). \tag{52}$$

Let us describe one of the possibilities to prove (52). Suppose that the statement (52) is not true. Then there are an $\epsilon > 0$ and a sequence $\psi_{m_k} \in \mathfrak{M}^{m_k}$ with $m_k \to \infty$ when $k \to \infty$, such that

$$\operatorname{dist}\{\psi_{m_k}, \mathfrak{M}\} \geqslant \epsilon \quad \text{for all} \quad \{m_k\}_{k=1}^{\infty}. \tag{53}$$

Due to (48) and (50) the sum $\bigcup_{m=1}^{\infty} \mathfrak{M}^m \cup \mathfrak{M} \equiv \mathbb{K}$ is a bounded set in $\mathscr{H}$ and therefore is a precompact in H_0. As $\mathfrak{M}$ is a B-attractor we can find such $T \in \mathbb{R}^+$ that

$$V_T(K) \subset O_{\epsilon/2}(\mathfrak{M}). \tag{54}$$

After this we choose $\phi_{m_k} \in \mathfrak{M}^{m_k}$ in such a way that $\psi_{m_k} = V_T^{m_k}(\phi_{m_k})$. It is possible because $\mathfrak{M}^{m_k}$ is invariant with respect to $V_t^{m_k}$. The sequence $\{\phi_{m_k}\}_{k=1}^{\infty}$ belongs to precompact $\mathbb{K}$ and in virtue of this has a limit point ϕ. Suppose for convenience that all ϕ_{m_k}, $k = 1, 2, \ldots$, converge to ϕ in H_0. Let us consider the sequence $\{v_k(t) \equiv V_t^{m_k}(\phi_{m_k})\}_{k=1}^{\infty}$ on the segment $t \in [0, T]$. We have for it the uniform estimates of §§2 and 3. Because of these and of (47_k) we can choose a subsequence $\{v_{k_j}(t),\, t \in [0, T]\}$ which converges to the solution $v(t, \phi)$ of the problem (1), (2), (4_k) for any $t \in [0, T]$ and particularly for $t = T$. So we have proved that $v_{k_j}(T) = \phi_{m_{k_j}}$, $j \to \infty$, converge to $v(T, \phi)$. But this fact contradicts to (53) since $v(T, \phi) \in O_{\epsilon/2}(\mathfrak{M})$.

There are some other finite-dimensional approximations of the problem (1), (2), (4_k) which are globally stable. Among them there are difference-differential and finite-difference approximations.

Appendix A

Let us explain why $\mathscr{H}$ is separable. For this purpose let us introduce the Banach space X. Its elements are the maps $U: \Omega \to \mathbb{R}^{21}$. The components of U we partition in three groups: to the first group come the elements u_k, $k = 1, 2, 3$, of $L_2(\Omega, \mathbb{R}^1)$, to the second group come the elements $u_{i(k)}$, $i, k = 1, 2, 3$, of $L_2(\Omega, \mathbb{R}^1)$ and to the third group, the elements $u_{(ik)}$, $i, k = 1, 2, 3$, of $L_{2+2\mu}(\Omega, \mathbb{R}^1)$. The norm of U in X is defined as

$$\|U\|_X = \sum_{k=1}^{3} \|u_k\|_{2,\Omega} + \sum_{i,k=1}^{3} \|u_{i(k)}\|_{2,\Omega} + \sum_{i,k=1}^{3} \|u_{(ik)}\|_{2+2\mu,\Omega}.$$

This space is separable because each space $L_m(\Omega, \mathbb{R}^1)$, $m \geqslant 1$, is separable. Our space $\mathscr{H}$ is a closed linear subset (i.e. subspace) of X and therefore $\mathscr{H}$ is also separable.

Appendix B. Some additional estimates

Let us prove that if $f \in H_0$ and $\phi \in \mathscr{H}$ then the solution $v(t, \phi)$ has the finite norm $\|v(t, \phi)\|_{\mathscr{H}}$ for any $t \in \mathbb{R}^+$. We restrict ourselves to the case when Ω is bounded and f does not depend on t. (The general case when f depends on t and Ω is arbitrary can be considered by the same method.) For such Ω we can use the relations (45_k).

Let us take the relation (15) and use it to get an inequality which is slightly different from (17):

$$\begin{aligned}
&\tfrac{3}{4}\|v_t(t)\|^2 + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathscr{B}(\hat{v}^2(x,t))\,\mathrm{d}x \\
&\quad \leqslant \gamma_p^2 \,\|\hat{v}(t)\|_{2+2\mu,\Omega} \,\|v(t)\|^{1-\alpha} \,\|v_x(t)\|^{\alpha} \,\|v_t(t)\|^{1-\alpha} \,\|v_{xt}(t)\|^{\alpha} + \|f\|^2 \\
&\quad \leqslant c_6 \,\|\hat{v}(t)\|_{2+2\mu,\Omega}^{1+\alpha} \,\|v_t(t)\|^{1-\alpha} \,\|v_{xt}(t)\|^{\alpha} + \|f\|^2
\end{aligned} \tag{55}$$

with $$c_6 = \gamma_p^2 (1/\sqrt{2}\,|\Omega|^{\mu/(2+\mu)})^\alpha \max\left\{R_0;\ \sup_{t\in\mathbb{R}^+} \|v(t)\|^{1-\alpha}\right\} = \Phi_9(\|v(0)\|).$$

Now we multiply this inequality by 2, sum it with (19) multiplied by t and majorize the right-hand side of resulting inequality using Young's inequality in the following way:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_\Omega \mathscr{B}(\hat{v}^2(x,t))\,\mathrm{d}x + t\,\|v_t(t)\|^2\right] + \tfrac{1}{2}\|v_t(t)\|^2 \\ + \nu_0(2-\alpha)\,t\,\|v_{xt}(t)\|^2 \leqslant 2c_6\,\|\hat{v}(t)\|^{1+\alpha}_{2+2\mu,\Omega}\,\|v_t(t)\|^{1-\alpha} \\ \times(\epsilon t\,\|v_{xt}(t)\|^2)^{\alpha/2}(\epsilon t)^{-\alpha/2} + c_1\,\|\hat{v}(t)\|^{1/(1-\alpha)}_{2+2\mu,\Omega}\,(t\,\|v_t(t)\|^2) \\ + 2\,\|f\|^2 \leqslant \tfrac{1}{2}\alpha\,\epsilon t\,\|v_{xt}(t)\|^2 + \tfrac{1}{2}(2-\alpha)(2c_6)^{2/(2-\alpha)}\,(\epsilon t)^{-\alpha/(\alpha-2)}\,\|\hat{v}(t)\|^{2(1+\alpha)/(2-\alpha)}_{2+2\mu,\Omega} \\ \times\|v_t(t)\|^{2(1-\alpha)/(2-\alpha)} + c_1\,\|\hat{v}(t)\|^{1/(1-\alpha)}_{2+2\mu,\Omega}\,(t\,\|v_t(t)\|^2) + 2\,\|f\|^2.\end{aligned} \tag{56}$$

According to (3_6)

$$\tfrac{1}{2}\nu_0\,\|\hat{v}(t)\|^2 + (\nu_1/2(1+\mu)]\,\|\hat{v}(t)\|^{2+2\mu}_{2+2\mu,\Omega} + t\,\|v_t(t)\|^2 \leqslant \frac{1}{2}\int_\Omega \mathscr{B}(\hat{v}^2(x,t))\,\mathrm{d}x + t\,\|v_t(t)\|^2 \equiv z(t) \tag{57}$$

and therefore from (56) with $\epsilon = \nu_0\,\alpha^{-1}(2-\alpha)$ we can get

$$\begin{aligned}\mathrm{d}z(t)/\mathrm{d}t + \tfrac{1}{2}\,\|v_t(t)\|^2 + \tfrac{1}{2}\nu_0(2-\alpha)\,t\,\|v_{xt}(t)\|^2 \\ \leqslant c_7\,t^{-\alpha/(2-\alpha)}\,\|\hat{v}(t)\|^{2(1+\alpha)/(1-\alpha)}_{2+2\mu,\Omega}\,(t\,\|v_t(t)\|^2)^{(1-\alpha)/(2-\alpha)}\,t^{-(1-\alpha)/(2-\alpha)} \\ + c_1\,\|\hat{v}(t)\|^{1/(1-\alpha)}_{2+2\mu,\Omega}\,z(t) + 2\,\|f\|^2 \leqslant c_8\,t^{-1/(2-\alpha)}\,z^{\gamma_1}(t) + c_9\,z^{\gamma_2}(t) + 2\,\|f\|^2,\end{aligned} \tag{58}$$

where

$$c_7 = (2-\alpha)\,2^{-1}(2c_6)^{2/(2-\alpha)}\,[\nu_0\,\alpha^{-1}(2-\alpha)]^{-\alpha/(2-\alpha)},\quad c_8 = c_7\,[\nu_1^{-1}(2+2\mu)]^{(1+\alpha)/(2-\alpha)(1+\mu)},$$
$$c_9 = c_1\,[\nu_1^{-1}(2+2\mu)]^{1/(1-\alpha)(2+2\mu)},\quad \gamma_1 = [1+\alpha+(1-\alpha)\,(1+\mu)]\,(2-\alpha)^{-1}(1+\mu)^{-1}$$

and

$$\gamma_2 = 1+(1-\alpha)^{-1}(2+2\mu)^{-1}.$$

The concrete values of c_k and γ_k are not very important. It is important only that the power $-1/(2-\alpha)$ is more than -1 for any $\mu \geqslant \frac{1}{4}$. The powers γ_1 and γ_2 do not exceed the number 2. All this permits to derive from (58) the inequality

$$\mathrm{d}z_1(t)/\mathrm{d}t \leqslant c_{10}(t)\,z_1^2(t), \tag{59}$$

where $$z_1(t) = 1+z(t)$$

and $$c_{10}(t) = c_8\,t^{-1/(2-\alpha)} + c_9 + 2\,\|f\|^2.$$

The integration of (59) gives

$$z_1(t) \leqslant z_1(0)/1 - z_1(0)\,c_{11}(t), \tag{60_1}$$

where $$c_{11}(t) = c_8(2-\alpha)(1-\alpha)^{-1}\,t^{(1-\alpha)/(2-\alpha)} + t(c_9+2\,\|f\|^2).$$

This inequality holds for $t\in[0,\tau_\phi)$ where τ_ϕ is a solution of the equation

$$c_{11}(\tau_\phi) = z_1^{-1}(0) = \left[1+\frac{1}{2}\int_\Omega \mathscr{B}(\hat{\phi}(x))\,\mathrm{d}x\right]^{-1},\quad \phi = v(0). \tag{60_2}$$

Using (3_6) it is not difficult to find a majorant Φ for τ_ϕ^{-1}:

$$\tau_\phi^{-1} \leqslant \Phi_{10}(\|\hat{\phi}\|_{2+2\mu,\Omega}). \tag{60_3}$$

From the inequality (60_1) follows for $t \in [0, \tau_\phi)$ the estimate

$$\tfrac{1}{2}\nu_0 \|\hat{v}(t,\phi)\|^2 + [\nu_1/(2+2\mu)] \|\hat{v}(t,\phi)\|_{2+2\mu,\Omega}^{2+2\mu} + t\|v_t(t,\phi)\|^2 \leqslant \Phi_{11}((t-\tau_\phi)^{-1}, \|\hat{\phi}\|_{2+2\mu,\Omega}). \tag{61}$$

The inequalities (60_1) and (58) give the next estimate

$$\int_0^t [\|v_\tau(\tau)\|^2 + \nu_0(2-\alpha)\tau\|v_{x\tau}(\tau)\|^2]\,\mathrm{d}\tau \leqslant \Phi_{12}((t-\tau_\phi)^{-1}, \|\hat{\phi}\|_{2+2\mu,\Omega}), \quad t \in [0,\tau_\phi). \tag{62}$$

The both estimates (61) and (62) are local in t. But they in a combination with the estimations (26), (27) for $t \geqslant \frac{1}{2}\tau_\phi$ give the non-local estimates

$$\nu_0 \|\hat{v}(t)\|^2 + [\nu_1/(1+\mu)] \|\hat{v}(t)\|_{2+2\mu,\Omega}^{2+2\mu} + t\|v_t(t)\|^2 \leqslant \Phi_{13}(t, \|\hat{v}(0)\|_{2+2\mu,\Omega}) \tag{63_1}$$

and

$$\int_0^t (\|v_\tau(\tau)\|^2 + \nu_0\tau\|v_{x\tau}(\tau)\|^2)\,\mathrm{d}\tau \leqslant \Phi_{14}(t, \|\hat{v}(0)\|_{2+2\mu,\Omega}). \tag{63_2}$$

Appendix C. On non-homogeneous boundary condition

Let us take instead of (4_1) the non-homogeneous boundary condition

$$v(t)|_{\partial\Omega} = \alpha|_{\partial\Omega}, \quad t \in \mathbb{R}^+, \tag{64}$$

and suppose that $\alpha|_{\partial\Omega}$ can be continued on Ω as a solenoidal field $\operatorname{rot} b(x)$, $x \in \Omega$, with the following properties:

$$b \in W^2_{2+2\mu}(\Omega), \quad \operatorname{rot} b|_{\partial\Omega} = \alpha|_{\partial\Omega}. \tag{65}$$

For such α we can prove a global unique solvability of the problem (1), (2), (64), (4_2) and the existence for it a compact minimal global B-attractor with a finite number of determining modes.

For this purpose we introduce the functions:

$$u(x,t,\epsilon,\rho) = v(x,t) - a(x,\epsilon,\rho),$$

where $a(x,\epsilon,\rho) = \operatorname{rot}(\zeta(x,\epsilon,\rho)\,b(x))$ and $\zeta(x,\epsilon,\rho)$ are very smooth scalar cut-functions constructed in §4, ch. V [4] and depending on parameters: $\epsilon \in (0,1]$, $\rho \in (0,\rho_0]$, $\rho_0 \ll 1$. The values of ζ belong to $[0,1]$ and $\zeta \equiv 1$ near $\partial\Omega$, so $a|_{\partial\Omega} = \alpha|_{\partial\Omega}$. For any $\epsilon_1 \in (0,1]$ we can choose the parameters ϵ and ρ such that the inequality

$$\left|\int_\Omega w_k a_i w_{ix_k}\,\mathrm{d}x\right| \leqslant \epsilon_1 \|w_x\|^2 \tag{66}$$

holds for any $w \in H_1$. We take $\epsilon_1 = \nu_0/16$ and fix (for all subsequent consideration) the corresponding ϵ and ρ so that

$$\left|\int_\Omega w_k a_{ik} w_i\,\mathrm{d}x\right| = 2\left|\int_\Omega w_k a_i w_{ix_k}\,\mathrm{d}x\right| \leqslant \frac{\nu_0}{16}\|\hat{w}\|^2, \tag{67}$$

where $\hat{w} = (w_{ik})$, $w_{ik} = w_{ix_k} + w_{kx_i}$, and $a_{ik} = a_{ix_k} + a_{kx_i}$, $i,k = 1,2,3$. The norms $\|a\|$,

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$\|\hat{a}\|$, $\|a\|_{2+2\mu,\Omega}$ do not exceed a constant c_{12}. The problem (1), (2), (64), (4_2) for v, q is equivalent to the following problem

$$u_{it}(t)+\sum_{k=1}^{3}(u_k(t)+a_k)(u_{ik}(t)+a_{ik})-\sum_{k=1}^{3}\frac{\partial}{\partial x_i}T_{ik}(\hat{u}(t)+\hat{a})$$
$$=-q_{x_i}(t)+f_i,\quad i=1,2,3,\quad (68_1)$$

$$\operatorname{div} u=0,\quad u|_{\partial\Omega}=0,\quad u|_{t=0}=\phi-a\equiv\psi,\quad (68_2)$$

for u, q. System (68_1) differs from system (1) by some low-order terms and external forces. We shall show that problem (68_k), $k=1,2$, has the same properties as problem (1), (2), (4_k), $k=1,2$. In particular it has a compact minimal global B-attractor in the phase space H_0 if T_{ik} have the form (3_1).

Let us prove the existence of a bounded B-absorbing set for problem (68_k), $k=1$, 2. For this we multiply (68_1) by u_i, take the sum over $i=1,2,3$, integrate over $x\in\Omega$ and transform the result by an integration by parts using the equations (68_2). It gives the 'energy' relation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+\frac{1}{2}\int_\Omega T_{ik}(\hat{v}(t))\,v_{ik}(t)\,\mathrm{d}x$$
$$=\frac{1}{2}\int_\Omega T_{ik}(\hat{v}(t))\,a_{ik}\,\mathrm{d}x-\int_\Omega(u_k(t)+a_k)\,a_{ik}\,u_i(t)\,\mathrm{d}x+(f,u(t)).\quad (69)$$

Using properties (i), (ii) and (67) we derive from (69) the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+\nu_0\|\hat{v}(t)\|^2+\nu_1\|\hat{v}(t)\|^{2+2\mu}_{2+2\mu,\Omega}$$
$$\leqslant\int_\Omega c_1(1+|\hat{v}(t)|^{2\mu})\,|\hat{v}(t)|\sum_{ik}|a_{ik}|\,\mathrm{d}x+\tfrac{1}{8}\nu_0\|\hat{u}(t)\|^2$$
$$+2\|\hat{a}\|\,\|a\|_{3,\Omega}\|u\|_{6,\Omega}+2\|f\|_{(-1)}\|u_x\|.\quad (70)$$

Due to (16) with $p=6$ and some elementary inequalities we get from (70) the following inequalities:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+\nu_0\|\hat{v}(t)\|^2+\nu_1\|\hat{v}(t)\|^{2+2\mu}_{2+2\mu,\Omega}$$
$$\leqslant 3c_1\int_\Omega(|\hat{v}(t)|+|\hat{v}(t)|^{1+2\mu})|\hat{a}|\,\mathrm{d}x+\tfrac{1}{8}\nu_0\|\hat{u}(t)\|^2$$
$$+\surd 2\gamma_6\|\hat{a}\|\,\|a\|_{3,\Omega}\|\hat{u}\|+\surd 2\|f\|_{(-1)}\|\hat{u}\|\leqslant 3c_1\|\hat{v}(t)\|\,\|\hat{a}\|$$
$$+3c_1\|\hat{v}(t)\|^{1+2\mu}_{2+2\mu,\Omega}\|\hat{a}\|_{2+2\mu,\Omega}+\tfrac{1}{8}\nu_0\|\hat{u}(t)\|^2+c_{13}\|\hat{u}(t)\|$$
$$\leqslant\nu_0\epsilon_1\|\hat{v}(t)\|^2+9c_1^2(4\nu_0\epsilon_1)^{-1}\|\hat{a}\|^2+\frac{1+2\mu}{2+2\mu}\epsilon_2^{(2+2\mu)/(1+2\mu)}\|\hat{v}(t)\|^{2+2\mu}_{2+2\mu,\Omega}$$
$$+\frac{1}{2+2\mu}(3c_1\epsilon_2^{-1})^{2+2\mu}\|\hat{a}\|^{2+2\mu}_{2+2\mu,\Omega}+\tfrac{1}{4}\nu_0\|\hat{u}(t)\|^2+2c_{13}^2\nu_0^{-1}$$
$$\equiv\nu_0\epsilon_1\|\hat{v}(t)\|^2+\frac{1+2\mu}{2+2\mu}\epsilon_2^{(2+2\mu)/(1+2\mu)}\|\hat{v}(t)\|^{2+2\mu}_{2+2\mu,\Omega}$$
$$+\tfrac{1}{4}\nu_0\|\hat{u}(t)\|^2+c_{14},\quad (71)$$

where

$$c_{13} = \sqrt{2}(\gamma_6 \|\hat{a}\| \, \|a\|_{3,\Omega} + \|f\|_{(-1)})$$

and

$$c_{14} \equiv c_{14}(\epsilon_1, \epsilon_2) = 9c_1^2(4\nu_0 \epsilon_1)^{-1} \|\hat{a}\|^2 + \frac{1}{2+2\mu}(3c_1 \epsilon_2^{-1})^{2+2\mu} \|\hat{a}\|^{2+2\mu}_{2+2\mu,\Omega} + 2c_{13}^2 \nu_0^{-1}.$$

Let us choose $\epsilon_1 = \frac{1}{8}\nu_0$ and ϵ_2 such that $[(1+2\mu)/(2+2\mu)]\,\epsilon_2^{(2+2\mu)/(1+2\mu)} = \frac{1}{2}\nu_1$. Then from (71) follows the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \tfrac{7}{8}\nu_0 \|\hat{v}(t)\|^2 + \tfrac{1}{2}\nu_1 \|\hat{v}(t)\|^{2+2\mu}_{2+2\mu,\Omega} \leqslant \tfrac{1}{4}\nu_0 \|\hat{u}(t)\|^2 + c_{14}. \tag{72}$$

It and inequality

$$\|\hat{v}(t)\|^2 \geqslant (1-\epsilon_3)\,\|\hat{u}(t)\|^2 - (\epsilon_3^{-1} - 1)\,\|\hat{a}\|^2, \quad \forall \epsilon_3 \in (0,1),$$

with $\epsilon_3 = \frac{1}{7}$ give the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \tfrac{1}{2}\nu_0 \|\hat{u}(t)\|^2 + \tfrac{1}{2}\nu_1 \|\hat{v}(t)\|^{2+2\mu}_{2+2\mu,\Omega} \leqslant c_{15}, \tag{73}$$

where

$$c_{15} = c_{14} + \tfrac{21}{4}\nu_0 \|\hat{a}\|^2.$$

From (73) follows (see (10)–(14_k)) that the balls $B_k(H_0)$ with $\forall R > R_0' = \sqrt{(c_{15}(\nu_0 \lambda_1)^{-1})}$ are B-absorbing sets for problem (68_k), $k = 1, 2$.

The inequality

$$\tfrac{1}{2}\nu_0 \int_0^t \|\hat{v}(\tau)\|^2 \,\mathrm{d}\tau + \tfrac{1}{2}\nu_1 \int_0^t \|\hat{v}(\tau)\,\|^{2+2\mu}_{2+\mu,\Omega} \,\mathrm{d}\tau \leqslant \|u(0)\|^2 + tc_{15} \tag{74}$$

is a consequence of (73). It and

$$\|\hat{u}(t)\|^{2+2\mu}_{2+2\mu,\Omega} \leqslant (\|\hat{v}(t)\|_{2+2\mu,\Omega} + \|\hat{a}\|_{2+2\mu,\Omega})^{2+2\mu} \leqslant 2^{2+2\mu}(\|\hat{v}(t)\|^{2+2\mu}_{2+2\mu,\Omega} + \|\hat{a}\|^{2+2\mu}_{2+2\mu,\Omega})$$

give the estimate

$$\nu_0 \int_0^t \|\hat{u}(\tau)\|^2 \,\mathrm{d}\tau + \nu_1 \int_0^t \|\hat{u}(\tau)\|^{2+2\mu}_{2+2\mu,\Omega} \,\mathrm{d}\tau \leqslant \Phi_{15}(t, \|u(0)\|). \tag{75}$$

The estimations of more stronger norms of $u(t) \equiv u(t, \bar{\Phi})$ (for T_{ik} of the form (3_1)) are derived as in §3. They have a local character and do not require in the special continuation of $\alpha|_{\partial\Omega}$ on Ω described in the beginning of this section, if we know the existence of a bounded B-absorbing set and estimate (73).

The proofs of a global unique solvability of problem (68_k), $k = 1, 2$, and the existence of a compact minimal global B-attractor for the problem are analogous to the proofs of these facts for problem (1), (2), (4_k), $k = 1, 2$.

Appendix D. On some more general case of $T(\hat{v})$

All results described above for the system (1) with $T_{ik}(\hat{v})$ having the form (3_1) are true for slightly more general $T_{ik}(\hat{v})$, namely for

$$T_{ik}(\hat{v}) = \partial\mathscr{D}(\hat{v})/\partial v_{ik}, \quad i,k = 1,2,3. \tag{76}$$

if $\mathscr{D}(\hat{v})$ (the potential of $T(\hat{v})$) satisfies the following conditions:

(*a*) $\mathscr{D}: M_s^{3\times 3} \to \mathbb{R}^1$ belongs to $W^2_{\infty,\mathrm{loc}}$ and $\mathscr{D}(0) = 0$;

(*b*) $\mathscr{D}(\hat{v}) \geqslant \nu_0 |\hat{v}|^2 + \nu_1 |\hat{v}|^{2+2\mu}$, $\mu \geqslant \frac{1}{4}$, $\nu_0 > 0$, $\nu_1 > 0$;

(*c*) $\left(\dfrac{\partial^2 \mathscr{D}(\hat{v})}{\partial \hat{v}^2}\zeta\right) : \zeta \equiv \dfrac{\partial^2 \mathscr{D}(\hat{v})}{\partial v_{ik}\, \partial v_{jl}} \zeta_{ik}\zeta_{jl} \geqslant \nu_2 |\zeta|^2 \equiv \nu_2 \Sigma_{ik} \zeta_{ik}^2$, $\nu_2 > 0$, for any $\hat{v}$ and ζ from $M_s^{3\times 3}$;

(*d*) $\left|\dfrac{\partial \mathscr{D}(\hat{v})}{\partial \hat{v}}\right| = \left[\Sigma_{ik}\left(\dfrac{\partial \mathscr{D}(\hat{v})}{\partial v_{ik}}\right)^2\right]^{\frac{1}{2}} \leqslant \nu_3 |\hat{v}|^{1+2\mu} + \nu_4$ for any $\hat{v} \in M_s^{3\times 3}$, where $M_s^{3\times 3}$ is the set of all symmetric matrices 3×3.

The proofs of these results are fully analogous to the proofs given in §§2–6.

Remark. In all our considerations the stress tensor $T(\hat{v}) = (T_{ik}(\hat{v}))$ is symmetric:

$$T_{ik}(\hat{v}) = T_{ki}(\hat{v}).$$

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